1301. The following identity holds for all $A, B \in \mathbb{R}$:

$$\cos(A+B) \equiv \cos A \cos B - \sin A \sin B.$$

Use this compound-angle formula to solve the equation $\cos(x+30°) = \sin x$, for $x \in [0, 360°)$.

1302. A quantum physicist is measuring the energy of electrons emerging from a scattering experiment. The speeds involved in the experiment are being increased linearly over time. The average energy, in Joules, of the scattered electrons is modelled as changing, over t measured in seconds, according to

$$\frac{dE}{dt} = 2.15616t - 0.088210t^2.$$

(a) Give the initial rate of change of E.

(b) Find, for each the following, the times $t \geq 0$ for which the relevant quantity is increasing. Give your answers in set notation.

i. energy,

ii. rate of change of energy.

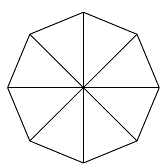
1303. Let A and B be events with non-zero probabilities. Show that A and B cannot be mutually exclusive and independent.

1304. Simplify the following:

(a) $2^{\log_2 x + \log_2 y}$,

(b) $e^{\ln x + \ln y}$.

1305. The area of a regular octagon of side length 1 is $A = 2(1+\sqrt{2})$. Use this result to find the exact distance from the centre to any vertex.



1306. Sketch a quadratic graph $y = \mathrm{h}(x)$ for which

$$\int_2^4 \mathrm{h}(x)\,dx = 1, \quad \mathrm{h}(2) = \mathrm{h}(4) = 0.$$

1307. A pump ejects water from a tank. Water, whose density is 1 kg per 1000 cm^3, accelerates from rest along a cylindrical pipe of length 50 cm and cross-sectional area 20 cm^2. It emerges with a velocity of 2 ms^{-1}. Acceleration is assumed to be constant.

(a) Find the total mass of water in the pipe.

(b) Find the acceleration of the water.

(c) Hence, find the force applied by the pump.

1308. Four lines are defined by equations $y = 2x - 3$, $y = 2x + 5$, $y = -2x - 3$, and $y = -2x + 5$. Show that the four lines enclose a rhombus.

1309. A sample $\{x_i\}$ has mean $\bar{x}$ and variance s_x^2. Write down the mean value of

(a) $x_i - \bar{x}$,

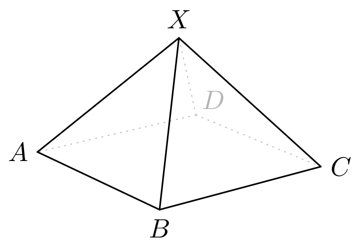
(b) $(x_i - \bar{x})^2 - s_x^2$.

1310. Functions f and g are such that $x = a$ is a root of $\mathrm{f}(x) = 0$, $x = b$ is a root of $\mathrm{g}(x) = 0$, and $x = c$ is a root of $\mathrm{f}(x) = \mathrm{g}(x)$. State, with a reason, whether the following hold:

(a) If $a = b$, then $a = b = c$.

(b) If $a = c$, then $a = b = c$.

1311. The square-based pyramid shown below is formed of eight edges of unit length.



Find the total surface area.

1312. Show that the diameter, from vertex to vertex, of a regular decagon of side length l is $l \operatorname{cosec} 18°$.

1313. A student writes: "Friction acts to oppose motion, so it can only go in one direction. Therefore, since Newton III has equal and opposite forces, friction can't obey Newton's third law." Explain carefully why this is incorrect.

1314. Solve for k in $\displaystyle\int_0^1 12x^2\,dx = \int_0^k x^3\,dx$.

1315. Write down the ranges of the following functions, when defined over the real numbers:

(a) $x \mapsto 2^x$,

(b) $x \mapsto 2^{-x}$,

(c) $x \mapsto -2^x$.

1316. The parabola $y = x^2 + px + q$ crosses the x axis at $x = a, b$. Write down the equation of the monic parabola which has intercepts at $x = -a, -b$.

1317. In this question, $\mathrm{lcm}(a, b)$ is the lowest common multiple of distinct natural numbers a, b. State whether each of the following is true or false:

(a) $\mathrm{lcm}(a, b) = ab$.

(b) If a and b are primes, then $\mathrm{lcm}(a, b) = ab$.

(c) If $\mathrm{lcm}(a, b) = ab$, then a and b are primes.

1318. Show that the normal to the curve $y = x^4 - x^2$ at $x = 1$ intersects the curve again.

1319. State, giving a reason, which of the implications $\Longrightarrow$, $\Longleftarrow$, $\Longleftrightarrow$ links the following statements concerning a real number x:

① $x \in [a, b]$,
② $x \in (a, b)$.

1320. You are given that the two lines $2x + py = 0$ and $x - (p+1)y = 4$ are perpendicular to one another. Determine all possible values of p.

1321. Write the following sets as lists of elements, in the form $\{a_1, a_2, ..., a_n\}$.

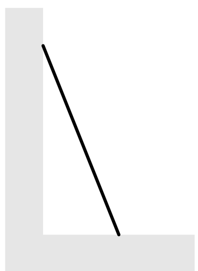
(a) $\{x \in \mathbb{Z} : x > 0\} \cap [-2, 2]$,
(b) $\{x \in \mathbb{Z} : x \le 0\} \cap (-3, 3]$,
(c) $\{x \in \mathbb{Z} : |x| \ge 2\} \cap (-4, 4)$.

1322. Prove that, if u_n is an arithmetic progression and f is a function with constant first derivative, then $\text{f}(u_n)$ is also an arithmetic progression.

1323. Rearrange $pt^6 + qt^3 + r = 0$ to make t the subject.

1324. A hypothesis test produces a p-value of 0.0421. A student writes: "Since the p-value is less than the significance level 5%, the null hypothesis is false." Explain what is wrong with this sentence.

1325. A ladder is placed on smooth horizontal ground, resting up against a rough vertical wall.



Prove that equilibrium cannot be maintained.

1326. Show that $2\sin^2 x + \cos x = 3$ has no real roots.

1327. The vertices V_i of a regular n-gon are at points (x_i, y_i), for $i = 1, 2, ..., n$, given, in radians, by

$$V_i : \left(2 + \sin \tfrac{\pi i}{20}, 4 - \cos \tfrac{\pi i}{20}\right).$$

Find the centre and number of sides of the n-gon.

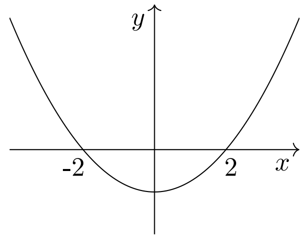
1328. Show that the curves $2y = x^2 + 1$ and $2x = y^2 + 1$ are tangent.

1329. A quadratic function h has $\text{h}(3) = -1$, $\text{h}'(3) = 0$, $\text{h}''(3) = -1$. Show that $\text{h}(x) = 0$ has no real roots.

1330. Solve $\ln x + \ln(4 + x) = \ln 2$.

1331. A line segment is drawn from the origin to a point (p, p^2). This line segment is then reflected in the line $y = p^2$, to form, with the y axis, an isosceles triangle. Show that this triangle has area p^3.

1332. The graph below is of $y = \text{g}'(x)$, for some cubic function g defined over the real numbers.



Find all possible cubic functions g.

1333. Solve $x + |x| > 1$, answering in set notation.

1334. Separate the variables in the following differential equation, writing it in the form $\text{f}(y)\frac{dy}{dx} = \text{g}(x)$ for some functions f and g:

$$\operatorname{cosec} x \frac{dy}{dx} - \tan y = 1.$$

1335. Determine the area of the smallest circle which completely encloses the curve $\frac{1}{9}x^2 + \frac{1}{16}y^2 = 25$.

1336. Find and correct the error in the following:

$$\int (2x+1)^2 \, dx = \tfrac{1}{3}(2x+1)^3 + c.$$

1337. Find the horizontal range of a projectile thrown from ground level, at an angle of 15° above the horizontal, at initial speed 70 ms^{-1}.

1338. Solve $\dfrac{1}{1+x} + \dfrac{1}{(1+x)^2} = 2$.

1339. Prove that, if a, b, c are positive numbers in GP with common ratio $r \neq 1$, then $a + c > 2b$. You may wish to consider the value of $a(1-r)^2$.

1340. In a biological study, the masses of 15 adult giant Pacific octopi are measured as having $\bar{m} = 13.6$ kg. Afterwards, it is discovered that the masses of four of the octopi were under-measured by 16%, due to a calibration error.

(a) Show that, with the information given above, the best estimate of the mean of the sample, to 3sf, is 14.3 kg.
(b) Why is your answer only an estimate?

1341. Simplify $\ln \dfrac{1}{e^x} + \ln(2e^x)$.

1342. Show that the x intercept of the line through (a, b) and (c, d) is at

$$x = \frac{bc - ad}{b - d}.$$

1343. Write down the functions whose derivatives are given by the following limits:

(a) $f'(x) = \lim\limits_{h \to 0} \dfrac{\sqrt{x + h + 1} - \sqrt{x - h + 1}}{2h}$.

(b) $g'(x) = \lim\limits_{a,b \to x} \dfrac{a^2 - b^2 + \frac{1}{a} - \frac{1}{b}}{a - b}$.

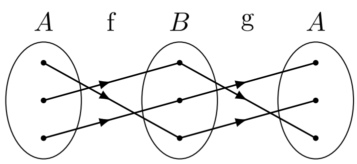
1344. An arithmetic progression begins s, s^2, s^2+6. Find all possible values for the hundredth term.

1345. Ignoring leap years etc., find the probability that, in any given week, at least one of a class of 25 students will have a birthday.

1346. Solve $\dfrac{1}{(1 + \sqrt{x})^2} + \dfrac{1}{(1 - \sqrt{x})^2} = 1$.

1347. Factorise the cubic $2x^3 - 5x^2 - 21x + 36$.

1348. Functions f and g map two sets A and B to each other, according to the following diagram. The two copies of A are ordered in the same way.



(a) Give the domains of fg and gf.

(b) Describe the behaviour of $x_{n+1} = gf(x_n)$.

1349. A four-sided die and a six-sided die are rolled at the same time. Find the probability that the score on the four-sided die is the larger of the two.

1350. Prove that $\tan^2 \theta \equiv \sec^2 \theta - 1$.

1351. A car of mass 400 kg is pulling a caravan of mass 800 kg up a slope of inclination 5°. Resistances of 250 N and 350 N act on car and caravan.

(a) Draw force diagrams for the vehicles.

(b) Determine the minimum driving force D such that the car can pull the caravan up a slope of unlimited length.

(c) Find the tension in the tow-bar with this D.

1352. Solve exactly the equation $x^{-\frac{1}{2}} + 1 = x^{\frac{1}{2}}$.

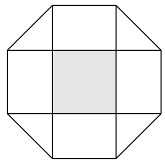
1353. The equation of a straight line, gradient m, passing through the point (p, q) is

$$\frac{y - q}{x - p} = m.$$

Sketch the following graphs:

(a) $\dfrac{|y - q|}{x - p} = m$, (b) $\dfrac{y - q}{|x - p|} = m$.

1354. A point is chosen at random on the interior of a regular octagon. Find the probability that it lies within the shaded square depicted.



1355. True or false?

(a) $\frac{d}{dx}(1 + 2x) = 2$,

(b) $\frac{d}{dy}(1 + 2y) = 2$,

(c) $\frac{d}{dx}(1 + 2y) = 2$.

1356. The *triangular numbers* are defined by the ordinal formula $T_n = \frac{1}{2}n(n + 1)$, for $n \in \mathbb{N}$. Prove that

$$T_{n+1} + T_n \equiv \big(T_{n+1} - T_n\big)^2.$$

1357. Show that $\displaystyle\int_0^{1000} \frac{\left(1 + \sqrt[3]{x}\right)^2}{\sqrt[3]{x}}\, dx = 9650$.

1358. By finding perpendicular bisectors, or otherwise, find the equation of the circle which passes through the points $A : (1, 0)$, $B : (7, -2)$ and $C : (17, 8)$.

1359. Simplify $\dfrac{1 - 9x^{\frac{5}{2}}}{1 - 3x^{\frac{5}{4}}}$.

1360. Find $\dfrac{dy}{dx}$ in terms of x, if $\dfrac{d}{dx}\big(4x^2 + y\big) = 0$.

1361. Two competitors are pulling against each other in a tug-of-war. Each pulls on one end of the same rope, and, when the rope has been displaced by 1 metre in either direction, the tug-of-war is won. The combined mass of both competitors and the rope is 200 kg. For $t \geq 0$, each competitor exerts a variable driving force horizontally on the ground: competitor 1 exerts $F_1 = 1480 - 20t$; competitor 2 exerts $F_2 = 1460 - 10t$.

(a) Show that competitor 1 gains initially, but fails to achieve victory.

(b) Find the time at which competitor 2 wins.

1362. Solve the equation $|x^2 - 3x| = |3x - 1|$.

1363. One of the following statements is true; the other is not. Identify and disprove the false statement.

① $\sin(\theta - \phi) = 0 \implies \theta - \phi = 0$,
② $\sin(\theta - \phi) = 0 \impliedby \theta - \phi = 0$.

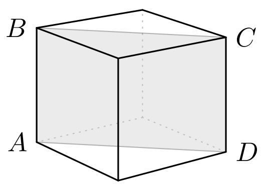
1364. Find the probability that, when two dice are rolled, the scores differ by more than 2.

1365. It is given that $x = \frac{4}{15}$ is a root of the expression $45x^3 - 27x^2 - 26x + 8$. Without using a calculator, factorise the expression fully.

1366. A line segment is defined by $\mathbf{r} = t\mathbf{i}$ for $t \in [-1, 1]$. Determine whether there are any points on this line segment which lie a distance $\sqrt{10}$ away from the point $(2, 3)$.

1367. If $\dfrac{d}{dt}\left(4x + 2t^2\right) = 6t$, find $\dfrac{dx}{dt}$ in terms of t.

1368. The diagram shows a cube of unit side length, with a rectangle formed between four vertices.



Three points are chosen at random inside the cube. Find the probability that no two of these points are separated by rectangle $ABCD$.

1369. State, giving a reason, which of the implications $\implies$, $\impliedby$, $\iff$ links the following statements concerning a polynomial function f:

① $\text{f}(x)$ has a factor of $(x - a)$,
② $\text{f}(x)$ has a root at $x = a$.

1370. A packing box of mass m is sitting on the floor of a delivery van, which is driving along a straight road. The coefficient of friction between the box and the floor of the van is μ. If the box is not to slide, determine, for the van,

(a) the maximum acceleration,
(b) the maximum deceleration.

1371. (a) Sketch, on one diagram, both triangles which have information $a = 5$, $b = 10$, $A = 25°$.
(b) Describe how these two possibilities emerge in
i. the sine rule, finding angle B,
ii. the cosine rule, finding length c.

1372. Calculate $9\displaystyle\sum_{i=1}^{\infty} \frac{1}{10^i}$.

1373. Show that the binomial distribution $X \sim B(5, 1/3)$ has two modal values.

1374. It is given that a polynomial graph $y = \text{h}(x)$, with domain $\mathbb{R}$, is concave everywhere.

(a) Explain why the graph can have a maximum of one stationary point.
(b) Hence, explain why the equation $\text{h}(x) = 0$ can have a maximum of two roots.

1375. A segment subtending an angle of $k\pi$ radians at the centre is marked on a circle with radius 2 cm. The area of the segment is $\frac{5}{3}\pi - 1$. Determine k, given that it is a rational number.

1376. Sketch $y = 2 - \sqrt[3]{x}$.

1377. Simultaneous equations are given as follows:

$$\begin{aligned} x + y - 2z &= -8, \\ 4x - 3y + z &= 6, \\ 2x + 5y - 2z &= -3. \end{aligned}$$

Solve to find x, y, z.

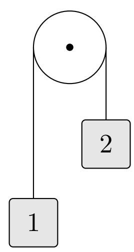
1378. Show that (x^2+1) is not a factor of $4x^3 - 12x^2 + 18$.

1379. A sequence has nth term $A_n = 1000 - 85n + 3n^2$. Determine the lowest value A_n in the sequence.

1380. State, with a reason, whether the following holds:

$$\frac{d}{dx}\int \text{f}(x)\,dx \equiv \int \text{f}'(x)dx.$$

1381. Two blocks are connected by a light, inextensible string, which is passed over a smooth, light, fixed pulley as shown in the diagram. Masses are given in kg. The system is released from rest.



(a) Find the time taken for the masses to move a distance of 1 metre.
(b) At this time, the string snaps. Find the total displacement of the 1 kg mass when it comes to instantaneous rest.

1382. Prove that no four distinct points on a cubic are collinear.

1383. Show that the Newton-Raphson iteration for the equation $x^3 + 3x - 1 = 0$ may be written as

$$x_{n+1} = \frac{2x_n^3 + 1}{3x_n^2 + 3}.$$

1384. True or false?

(a) Every cubic has a linear factor,

(b) Every quartic has a linear factor,

(c) Every quintic has a linear factor.

1385. Vectors **a** and **b** satisfy the following equations:

$$\mathbf{a} + \mathbf{b} = 2\mathbf{i} + 6\mathbf{j},$$
$$\mathbf{a} - 2\mathbf{b} = 8\mathbf{i} - 6\mathbf{j}.$$

Show that **a** and **b** are perpendicular.

1386. By considering either a unit circle or graphs, prove that $\sin^2(\theta + 180°) + \cos^2(\theta - 180°) \equiv 1$.

1387. If $\log_q p = x$, write $\left(\sqrt[3]{q}\right)^x$ in terms of p.

1388. Three white counters and three black counters are placed in a bag. Three of the counters are then drawn out. Find the probability of drawing out

(a) three white counters,

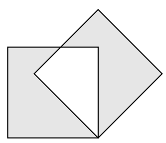
(b) two of one type and one of the other.

1389. Solve $\displaystyle\sum_{i=1}^{2} \frac{1}{1 - x^i} = 0$.

1390. The circles $(x-b)^2 + y^2 = b^2$ and $x^2 + y^2 = 1$ have at least one point of intersection. Find all possible values of b.

1391. An *Egyptian fraction* is a sum of fractions of the form $\frac{1}{n_i}$, for $n_i \in \mathbb{N}$. Write $\frac{5}{12}$ in this form.

1392. Two unit squares are placed as depicted below. All acute angles in the diagram are 45°.



Determine the exact area of the shaded region.

1393. State, with justification, whether the curve $y = |x|$ intersects the following curves:

(a) $y = |x + 1|$,

(b) $y = |x| + 1$.

(c) $y = |x + 1| + 1$.

1394. State, with a reason, whether the following gives a well-defined function:

$$\text{f} : \begin{cases} [-1, 1] \mapsto [0, 2.7] \\ x \mapsto \sqrt{1 - x^2}. \end{cases}$$

1395. Describe all functions f for which f$''$ is linear.

1396. A tank in the shape of a cuboid has dimensions $a \times a \times b$. Its volume is 1 cubic metre.

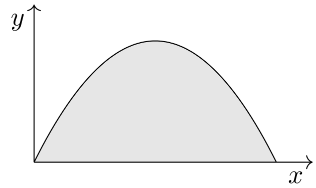
(a) Show that the surface area is given by

$$A = 2a^2 + 4a^{-1}.$$

(b) Hence, show that the minimum value of the surface area is 6 square metres.

1397. The parametric curve below has equations

$$x = 2t, \qquad y = 4t - 4t^2.$$



(a) Find the values t_1, t_2 at which $y = 0$.

(b) Write down the value of $\dfrac{dx}{dt}$.

(c) Find the area of the shaded region, using the parametric integration formula

$$A = \int_{t_1}^{t_2} y \frac{dx}{dt}\, dt.$$

1398. State, with a reason, whether the following holds: "It is possible to have, at a single point, friction and reaction exerted by the same object A, on the same object B, such that the two forces have a component in the same direction."

1399. By writing over base a, prove that $x^{\log_a y} \equiv y^{\log_a x}$.

1400. In applying for a course, candidates undertake two rounds of testing. Only candidates who pass the first test then sit the second. The probability that any given candidate passes the first test is 0.4, and the probability that any given candidate passes both tests is 0.1.

(a) Draw a tree diagram, finding all of the branch probabilities.

(b) Given that a particular candidate has not been accepted, find the probability that the failure occurred in the first test.

(c) Find the probability that, of two randomly chosen candidates, exactly one is accepted.

—— End of 14th Hundred ——